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Monodromy transform approach to solution of the Ernst equations in General Relativity

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Abstract. The approach, referred to as "monodromy transform", provides some general base for solution of all known integrable space - time symmetry reductions of Einstein equations for the case of pure vacuum gravitational fields, in the presence of gravitationally interacting massless fields, as well as for some string theory induced gravity models. In this communication we present the key points of this approach, applied to Einstein equations for vacuum and to Einstein - Maxwell equations for electrovacuum fields in the cases, reducible to the known Ernst equations. Definition of the monodromy data, formulation and solution of the direct and inverse problems of the monodromy transform, a proof of existence and uniqueness of their solutions, the structure of the basic linear singular integral equations and their regularizations, which lead to the equations of (quasi-)Fredholm type are also discussed. A construction of general local solution of these equations is given in terms of homogeneously convergent functional series.

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1 Introduction

The "monodromy transform" approach [1, 2, 3] is called so by some analogy with the names "scattering transform" or "spectral transform", used sometimes for the inverse scattering methods. This approach, based on the experience of various previously developed approaches [a], on the application of some powerful ideas of the modern theory of integrable systems and on a detail analysis of the proper internal structure of space-time symmetry reduced Einstein equations, provides the most general base for solution of various *integrable* reductions of Einstein equations, e.g., of vacuum and electrovacuum cases, reducible to the known Ernst equations (in both, hyperbolic and elliptic types). Without any change of basic elements, this approach is applicable to solution of space-time symmetry reduced Einstein equations for various pure gravitationally interacting massless fields (electromagnetic and Weyl spinor fields, scalar field and stiff matter fluid) as well as for some string theory induced gravity models [4].

[a]Avoiding a detail citation, we refer the readers to the references in a few papers cited here, but mainly – to a large and very useful F.J.Ernst's collection of related references and abstracts, accessible throw http://pages.slic.com/gravity

The first step of this construction is a definition for any local solution of the field equations of some set of functional parameters – *the monodromy data* for the corresponding solution of associated spectral problem, which role is similar to that, playing by the scattering data in the inverse scattering methods. (It is remarkable, that many properties of solutions can be expressed immediately in terms of the analytical structure of the corresponding monodromy data and that the Einstein equations imply trivial evolution for these data: they are functions of the spectral parameter only, independing of space - time coordinates.) The second step is a consideration of such transformation of "coordinates" in the entire space of local solutions from the field potentials to the corresponding monodromy data functions, i.e. formulation of the *direct* and *inverse problems* of this *monodromy transform.* In particular, it turns out, that there exists a linear singular integral equation with a scalar kernel, which solves the mentioned above inverse problem of a reconstruction of solutions of the Einstein equations from the monodromy data. This equation admits various equivalent regularizations which lead to linear integral equations of (quasi-) Fredholm type, for which the existence and uniqueness of solutions is proven easily for arbitrary choice of the monodromy data. This approach and the mentioned above integral equations, being equivalent integral equation forms of reduced Einstein equations (in particular, of vacuum or electrovacuum Ernst equations), admit various applications, such as the construction of various classes of exact solutions, which extend considerably the known classes of soliton solutions, exact linearization of various boundary value problems – the Cauchy problem or characteristic initial value problem and, probably, some others.

In this communication the key points of this construction are described for Einstein equations for vacuum and Einstein - Maxwell equations for electrovacuum fields with space-time symmetries, reducible to the Ernst equations. Definition of the monodromy data, formulation of the direct and inverse problems of our monodromy transform, the structure of the basic linear singular integral equations and their regularizations – the equations of (quasi-)Fredholm type are also discussed. A general local solution of these equations is given in terms of homogeneously convergent functional series.

2 Direct problem of the monodromy transform

An equivalent substitute for the study of the Ernst equations in both, hyperbolic or elliptic cases can be the analysis of the "spectral" problem for complex 3×3 matrices

$$\mathbf{\Psi}(\xi,\eta,w),\quad \mathbf{U}(\xi,\eta),\quad \mathbf{V}(\xi,\eta),\quad \mathbf{W}(\xi,\eta,w), \tag{1}$$

which are functions of two real or complex conjugated space-time coordinates ξ, η and a "spectral" parameter $w\in\bar{C}$ and which should satisfy two groups of conditions:

$$\left\{\begin{array}{l} 2i(w-\xi)\partial_\xi\mathbf{\Psi}=\mathbf{U}\cdot\mathbf{\Psi}\\ 2i(w-\eta)\partial_\eta\mathbf{\Psi}=\mathbf{V}\cdot\mathbf{\Psi}\end{array}\right. \left\Vert \begin{array}{l}\mathrm{rank}\,\mathbf{U}=1,\ \mathrm{tr}\,\mathbf{U}=i,\\ \mathrm{rank}\,\mathbf{V}=1,\ \mathrm{tr}\,\mathbf{V}=i.\end{array}\right\Vert\ \mathbf{\Psi}(\xi_0,\eta_0,w)=\mathbf{I} \tag{2}$$

where (ξ_0,η_0) are the coordinates of some chosen "reference" point, and

$$\left\{\begin{array}{l}\mathbf{\Psi}^\dagger\cdot\mathbf{W}\cdot\mathbf{\Psi}=\mathbf{W}_0(w)\\ \mathbf{W}_0^\dagger(w)=\mathbf{W}_0(w)\end{array}\right.\ \left\Vert\ \frac{\partial\mathbf{W}}{\partial w}=4i\mathbf{\Omega},\ \mathbf{\Omega}=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\ W^{55}=1.\right. \tag{3}$$

Here "†" is a Hermitian conjugation: $\mathbf{\Psi}^\dagger(\xi,\eta,w) \equiv \overline{\mathbf{\Psi}(\xi,\eta,\overline{w})}^T$; $\mathbf{W}_0(w)$ is an arbitrary Hermitian matrix function, depending on w only; the rows and columns of 3×3 - matrices are labeled by $3,4,5$, so that W^{55} is the lowest right component of $\mathbf{W}$. There exists [2] a one-to-one correspondence between the solutions of the Ernst equations and of the spectral problem (1) - (3). In particular, for the Ernst potentials we have identifications: $\partial_\xi\mathcal{E} = -\mathbf{U}_3{}^4$, $\partial_\eta\mathcal{E} = -\mathbf{V}_3{}^4$ and $\partial_\xi\Phi = \mathbf{U}_3{}^5$, $\partial_\eta\Phi = \mathbf{V}_3{}^5$.

A detail analysis [1, 2] of the analytical structure of solutions of (1) - (3) on the spectral plane shows the existence of some universal properties of $\mathbf{\Psi}(\xi,\eta,w)$. In particular it is holomorphic function of w everywhere outside four branchpoints and the cut $L = L_+ + L_-$ joining these points, as it is shown on Fig. 1. Nextly it turns out, that the behaviour of $\mathbf{\Psi}$ near the branchpoints can be described by so called *monodromy matrices* $\mathbf{T}_\pm(w)$, which characterise the linear transformations of the solution $\mathbf{\Psi}$ of the linear system (2), continued analytically along the paths $T_\pm$, rounding one of the branchpoints and joining different edges of L_+ or L_- respectively:

$$\mathbf{\Psi} \quad \xrightarrow{T_\pm} \quad \widetilde{\mathbf{\Psi}} = \mathbf{\Psi}\cdot\mathbf{T}_\pm(w), \qquad \mathbf{T}_\pm(w) = \mathbf{I} - 2\frac{\mathbf{l}_\pm(w)\otimes\mathbf{k}_\pm(w)}{(\mathbf{l}_\pm(w)\cdot\mathbf{k}_\pm(w))}. \tag{4}$$

It is remarkable, that these matrices, satisfying the identities $\mathbf{T}^2_\pm(w) \equiv \mathbf{I}$, are independent of the space-time coordinates ξ,η. The structure (4) allows to express $\mathbf{T}_\pm$ in terms of the four complex projective vectors $\mathbf{k}_\pm(w)$ and $\mathbf{l}_\pm(w)$, but it was found [1] that the conditions (3) relate unambiguously $\mathbf{l}_\pm(w)$ and $\mathbf{k}^\dagger_\pm(w)$ with the same suffices. Therefore, all components of $\mathbf{T}_\pm$ are determined completely by four scalar functions, which parametrise the components of two projective vectors $\mathbf{k}_\pm(w)$:

$$\mathbf{k}_\pm(w) = \{1\,,\mathbf{u}_\pm(w)\,,\,\mathbf{v}_\pm(w)\,\} \tag{5}$$

The functions $\mathbf{u}_\pm(w)$, $\mathbf{v}_\pm(w)$, which domains of holomorphicity are shown on Fig. 1, father are referred to as monodromy data (we note here, that for vacuum $\mathbf{v}_\pm(w)\equiv 0$), and the described above method for calculation of these functions for any solution of the Ernst equations solves a direct problem of our monodromy transform.

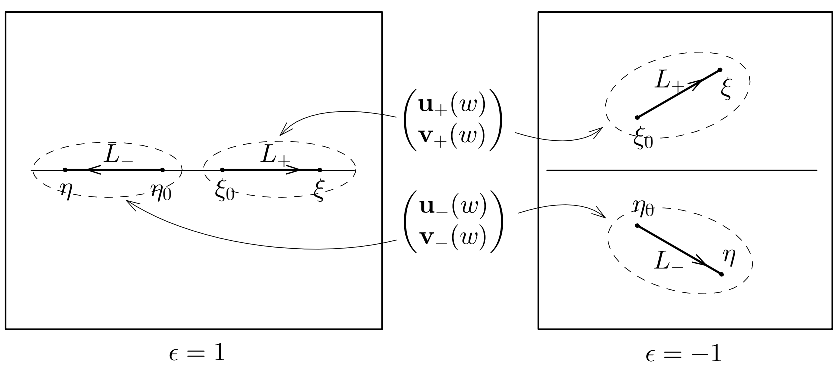

Fig. 1 The singular points, the structure of the cut $L = L_+ + L_-$ on the spectral plane and the domains, where the monodromy data functions $\mathbf{u}_\pm(w)$ and $\mathbf{v}_\pm(w)$ are defined and holomorphic, are shown here for the hyperbolic ($\epsilon = 1$) and elliptic ($\epsilon = -1$) cases separately.

3 Inverse problem of the monodromy transform

The statements, proved in [1, 2], imply the existence of two equivalent forms of the linear singular integral equation, which are equivalent to the Ernst equations:

$$-\frac{1}{\pi i}\int\limits_L \frac{[\lambda]_\zeta \mathcal{H}(\tau,\zeta)}{\zeta-\tau}\,\varphi(\xi,\eta,\zeta)\,d\,\zeta = \mathbf{k}(\tau), \qquad -\frac{1}{\pi i}\int\limits_L \frac{[\lambda^{-1}]_\zeta \mathcal{H}(\zeta,\tau)}{\zeta-\tau}\,\psi(\xi,\eta,\zeta)\,d\,\zeta = \mathbf{l}(\tau),$$

where $[\lambda]_\zeta$ is the jump of $\lambda=\sqrt{(w-\xi)(w-\eta)/(w-\xi_0)(w-\eta_0)}$, with $\lambda(\xi,\eta,\infty)=1$, at the point $\zeta\in L$ and $\mathcal{H}(\tau,\zeta)\equiv(\mathbf{k}(\tau)\cdot\mathbf{l}(\zeta))$. Vector solutions $\varphi(\xi,\eta,\tau)$, $\psi(\xi,\eta,\tau)$ of these equations together with the corresponding monodromy data vectors $\mathbf{k}$, $\mathbf{l}$ determine completely the solution of our spectral problem and hence, of the Ernst equations.

These equations admit equivalent regularizations by standard methods (here - the "left regularization"), which lead to equivalent equations of a (quasi-)Fredholm type:

$$\phi(\tau)+\int\limits_L \mathcal{F}(\tau,\zeta)\,\phi(\zeta)\,d\,\zeta=\mathbf{h}(\tau), \qquad \psi(\tau)+\int\limits_L \widetilde{\mathcal{F}}(\tau,\zeta)\,\psi(\zeta)d\,\zeta=\widetilde{\mathbf{h}}(\tau), \tag{6}$$

with $\phi(\tau)=-\mathcal{H}(\tau,\tau)\varphi(\tau)$ and the coefficients, determined by monodromy data:

$$\begin{aligned}
\mathcal{F}(\tau,\zeta)&=[\lambda]_\zeta\frac{1}{i\pi}\int\limits_L\frac{[\lambda^{-1}]_\chi}{\chi-\tau}\,\mathcal{S}(\chi,\zeta)\,d\,\chi \qquad & \mathbf{h}(\tau)&=\frac{1}{i\pi}\int\limits_L\frac{[\lambda^{-1}]_\chi}{\chi-\tau}\,\mathbf{k}(\chi)\,\mathbf{d}\,\chi\\
\widetilde{\mathcal{F}}(\tau,\zeta)&=-[\lambda^{-1}]_\zeta\frac{1}{i\pi}\int\limits_L\frac{[\lambda]_\chi}{\chi-\tau}\,\mathcal{S}(\zeta,\chi)\,d\,\chi \qquad & \widetilde{\mathbf{h}}(\tau)&=-\frac{1}{i\pi}\int\limits_L\frac{[\lambda]_\chi}{\chi-\tau}\,\frac{\mathbf{l}(\zeta)}{\mathcal{H}(\zeta,\zeta)}\,d\,\chi,
\end{aligned}$$

where $\mathcal{S}(\tau,\zeta)=\dfrac{\mathcal{H}(\tau,\zeta)-\mathcal{H}(\zeta,\zeta)}{i\pi(\zeta-\tau)\mathcal{H}(\zeta,\zeta)}$. The local solution of each of the equations (6) for any choice of monodromy data can be constructed by the known iterative method:

$$\begin{aligned}
&\phi(\tau)=\phi_0(\tau)+\sum_{n=1}^{\infty}\left(\phi_n(\tau)-\phi_{n-1}(\tau)\right),\\
&\phi_0(\tau)=\mathbf{h}(\tau), \qquad \phi_n(\tau)=\mathbf{h}(\tau)-\int\limits_L\mathcal{F}(\tau,\zeta)\phi_{n-1}(\zeta)\,d\,\zeta
\end{aligned}\tag{7}$$

A homogeneous convergency of these series, easily proven for local solutions, when ξ, η are close enough to their initial values ξ_0, η_0, provides the existence and uniqueness of solution of each of the discussed above equivalent integral equations for arbitrary chosen monodromy data. This means eventually that each of these equations solves the inverse problem of the monodromy transform for the Ernst equations.

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References

[1] G.A. Alekseev, Sov. Phys. Dokl. (USA) **30** (1985) 565
[2] G.A. Alekseev, Proceedings of the Steklov Inst. of Math. **3** (1988) 215
[3] G.A. Aleksejev, Proceedings of 8th Int. Workshop NEEDS'92, World Scientific 1992, p.5
[4] G.A. Alekseev, gr-qc/9911045, submitted to World Scientific.